

# CATEGORY THEORY

## CATEGORY I - POSETS

PAUL L. BAILEY

### 1. POSETS

**Definition 1** (Objects). A *poset*  $(P, \leq)$  consists of a set  $P$  together with a relation  $\leq$  on  $P$  with the following properties:

- (P1)  $a \leq a$  (Reflexivity)
- (P2)  $a \leq b$  and  $b \leq a$  implies  $a = b$  (Antisymmetry)
- (P3)  $a \leq b$  and  $b \leq c$  implies  $a \leq c$  (Transitivity)

The relation  $\leq$  is called a *partial order* on  $P$ , and the word poset is short for “partially ordered set”.

A partial order is a *total order* (or *linear order*) if, additionally, the relation satisfies

- (P4)  $a \leq b$  or  $b \leq a$  (Definiteness)

A set together with a total order is called a “totally ordered set”.

There are three archetypical examples of posets.

**Example 1.** Consider the set  $\mathbb{R}$  of real numbers, and its standard order relation  $\leq$ . Then  $\leq$  is a total order on  $\mathbb{R}$ , and  $(\mathbb{R}, \leq)$  is a poset. In fact, it is a totally ordered set.

**Example 2.** Let  $a, b \in \mathbb{Z}$ . We say that  $a$  *divides*  $b$ , and write  $a \mid b$ , if there exists  $k \in \mathbb{Z}$  such that  $b = ka$ .

Let  $P$  be a set of positive integers. Then divisibility is a partial order on  $P$ , and  $(P, \mid)$  is a poset.

**Example 3.** Let  $A$  and  $B$  be sets. We say that  $A$  is *contained in*  $B$ , and write  $A \subset B$ , if every element in  $A$  is an element in  $B$ .

Let  $\mathcal{C}$  be a collection of sets. Then containment is a partial order on  $\mathcal{C}$ . Thus,  $(\mathcal{C}, \subset)$  is a poset.

**Definition 2** (Subobjects). Let  $(P, \leq)$  be a poset. If  $Q \subset P$ , the restriction of  $\leq$  to  $Q$  satisfies the properties of a partial order on  $Q$ , making  $(Q, \leq)$  a poset. We may call  $(Q, \leq)$ , or just  $Q$ , a *subposet*.

**Definition 3** (Morphisms). Let  $P$  and  $Q$  be posets. A function  $f : P \rightarrow Q$  is called *order preserving* if

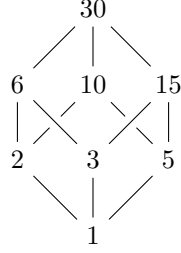
$$p_1 \leq p_2 \quad \Rightarrow \quad f(p_1) \leq f(p_2).$$

The identity map on  $P$  is order preserving, and the composition of order preserving functions is order preserving. Thus, posets with order preserving maps form a category.

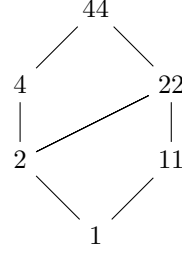
## 2. FACTOR SETS

**Definition 4** (Objects). Let  $n$  be a positive integer. The *factor set* of  $n$ , denoted  $\mathcal{F}(n)$ , is the set of all positive integers which divide  $n$ . Factor sets are partially ordered by divisibility.

Finite posets may be drawn using *Hasse diagrams*, as follows.



Factor Set of  $n = 30$



Factor Set of  $n = 44$

**Definition 5** (Morphisms). Let  $m$  and  $n$  be positive integers. A *morphism* from  $\mathcal{F}(n)$  to  $\mathcal{F}(m)$  is a function

$$f : \mathcal{F}(n) \rightarrow \mathcal{F}(m)$$

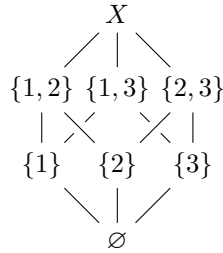
such that,

- (a)  $f(1) = 1$ ;
- (b)  $a \mid b \Rightarrow f(a) \mid f(b)$ , for all  $a, b \in \mathcal{F}(n)$ .

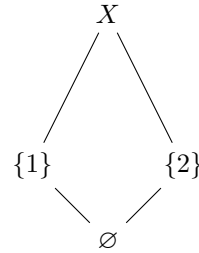
Factor sets form a nonfull subcategory of the category of partially ordered sets.

## 3. POWER SETS

**Definition 6** (Objects). Let  $X$  be any set. The *power set* of  $X$ , denoted  $\mathcal{P}(X)$ , is the set of all subsets of  $X$ . Power sets are partially ordered by containment.



Power set of  $X = \{1, 2, 3\}$



Power set of  $X = \{1, 2\}$

**Definition 7** (Morphisms). Let  $X$  and  $Y$  be sets. A *morphism* from  $\mathcal{P}(X)$  to  $\mathcal{P}(Y)$  is a function

$$f : \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$$

such that,

- (a)  $f(\emptyset) = \emptyset$ ;
- (b)  $A \subset B \Rightarrow f(A) \subset f(B)$ , for all  $A, B \subset X$ .

Power sets form a nonfull subcategory of the category of partially ordered sets.